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OFFICE OF

PROGRESS REPORT.

1. The following books have been purchased for the Library :—

1. *The Development of Mathematics in China and Japan* by Y. Mikami, B. G. Teubner, Dresden, Leipzig 1913. Price Rs. 16-10.

2. *Euclid's Books on Divisions of Figures* by Prof. R.C. Archibald, Ph. D. Cambridge University Press 1915. Price Rs. 5-4.

2. The following gentleman has been elected a *Life Member* of our Society—

Mr. C. P. Ramaswami Aiyar, B.A., B.L.—Vakil, High Court, and Fellow, Madras University, Madras.

3. The following gentlemen have been elected members of the Society.

1. *Mr. S. Ramanujan*—at present studying at Cambridge, (at concessional rate):
2. *Mr. F. H. V. Gulasegaram, B.A.*—Chief Lecturer in Mathematics, St. John's College, "Sangli Tope," Nallur, Jaffna (Ceylon).
3. *Mr. J. M. Bose, M.A., B.Sc., (Edin), Bar at Law*—Professor of Mathematics, Govt. Coll., Chittagong, Bengal.

4. The audited balance sheet of the Accounts for the year 1915, was published with the last issue of our Journal. The Budget for the current year is published herewith for the information of the General Body. It is a matter for congratulation that both the last year's balance and the present budget are satisfactory.

POONA,
31st March 1916.

D. D. KAPADIA,
Hony. Joint Secretary.

Budget for 1916.

Receipts—	Rs.	A.	P.	*Charges*—	Rs.	A.	P.
Opening Balance ...	1,783	11	10	Books and Journals ...	400	0	0
Subscriptions from members ...	1,350	0	0	Library charges ...	350	0	0
Sale of Journals ...	200	0	0	Printing Journal ...	600	0	0
Miscellaneous ...	50	0	0	Ordinary working expenses ...	250	0	0
				Closing Balance ...	1,783	11	10
Total ...	3,383	11	10	Total ...	3,383	11	10

MADRAS,
16th February 1916.

S. NARAYANA AIYAR, M.A., F.S.S.,
Hony. Treasurer.

Stability and Oscillations of Plane Kites.

By J. M. Bose, M.A., B.Sc.

1. In the Bulletin of the Calcutta Mathematical Society, Vol. II Pt. 1, I showed that the investigation of the stability and small oscillations of a plane kite, can be reduced to that of the roots of a certain cubic. But the results obtained in that paper were based on the assumption that the tension of the string acted at a point on the axis of symmetry, whose distance h from the origin remained constant during the small motions which were impressed. It has recently been pointed out to me by Prof. Bryan, that that this condition does not as a rule hold good, and if the variation of h be taken into account, the condition of stability would be somewhat different from those given in that paper.

In the present paper it is proposed to remove this restriction, and to give a more general investigation of the conditions of stability of plane kites.

Take the centre of gravity of the kite as origin, and as moving axes take the line joining the "head" and the "tail" (*i.e.*, the axis of symmetry, as y-axis, and a line lying in the plane of the kite as x axis, and finally a line perpendicular to them (*i.e.*, perpendicular to the plane of the kite) as z-axis. The axes are thus fixed relatively to the body of the kite and are moving with it.

Let A, B, C be the moments of inertia, $u, v, w, \theta_1, \theta_2, \theta_3$, the linear and angular velocities of the kite. The products of inertia evidently vanish since the kite is assumed to be a plane with an axis of symmetry.

Let us suppose that initially the plane of the kite is vertical; and let its plane be rotated round the axis of symmetry through an angle ψ, this brings the axis Oz to the position Oz_1; in this position let the plane of the kite be again rotated round Oz_1 through an angle θ, and finally let the plane of the kite be rotated round the x-axis (in its new position) through an angle ϕ.

The generalised angular co-ordinates θ, ϕ, ψ, are connected with the angular velocities $\theta_1, \theta_2, \theta_3$ by the following Euler's geometrical equations

$$\theta_1=\dot{\phi}+\dot{\psi}\sin\theta$$

$$\theta_2=\dot{\theta}\sin\phi+\dot{\psi}\cos\theta\cos\phi$$

$$\theta_3=\dot{\theta}\cos\phi-\dot{\psi}\cos\theta\sin\phi.$$

The impressed forces on the kite are

(i) the air resistance which (in the case of planes,) may be taken to be a single force R, acting at a point on the axis of symmetry called the centre of pressure. Let its co-ordinates be $(0, \eta, 0)$.

(ii) the weight W acting at the centre of gravity; its direction cosines with reference to the moving axes being $\sin\theta$, $\cos\theta\cos\phi$, $\cos\theta\sin\phi$.

(iii) the tension T of the string, acting in directions whose cosines are l, m, n at the instant under consideration. As a rule, the string bifurcates at a point, and the two parts are attached to two different points on the axis of symmetry, but we can always replace the two tensions by their resultant. Let the co-ordinates of the point of application of this resultant be (f, h, o).

We shall for the present suppose the string to be very long (almost infinite) so that during the small oscillations of the kite the direction of the string in space, as well as the magnitude of the tension, may both be assumed to be absolute constants.* The case of *finite* string will be dealt with afterwards.

If χ be the inclination of the string to the vertical, supposed constant, then it can be easily proved from considerations of elementary solid geometry, that the direction and point of application of the tension are given by

$$l=-(\cos\chi\sin\theta+\sin\chi\sin\psi\cos\theta)$$
$$m=-\cos\chi\cos\theta\cos\phi+\sin\chi(\cos\psi\sin\phi+\sin\psi\sin\theta\cos\phi)$$
$$n=-\cos\chi\cos\theta\sin\phi+\sin\chi(\cos\psi\cos\phi-\sin\psi\sin\theta\sin\phi)$$
$$f=h_o\frac{\sin(\overline{\phi+\chi})}{\sin\chi}\left(\sin\theta+\frac{l}{n}\cos\theta\sin\phi\right)$$
$$h=h_o\frac{\sin\overline{\phi+\chi}}{\sin\chi}\left(\cos\theta\cos\phi+\frac{m}{n}\cos\theta\sin\phi\right)$$

h_o being the initial value of h, and the quantities l, m, n, f, h are referred to the axes in their new position.

Let X, Y, Z be the components of the total force, L, M, N their moments about the axes, and T the constant tension of the string, then we have

$$X=Tl+W\sin\theta$$
$$Y=Tm-W\cos\theta\cos\phi$$
$$Z=Tn+W\cos\theta\sin\phi-R$$
$$L=hTn-\eta R$$
$$M=-fTn$$
$$N=T(fm-hl).$$

* I am indebted to Prof. Bryan for this suggestion.

In the position of equilibrium let the plane of the kite make an angle ϕ_o with the vertical, *i.e.*, the plane of the kite is rotated through an angle ϕ_o round the x-axis. Since the position of the old x—axis is unchanged, we have initially $\theta=0$, $\psi=0$, $\phi=\phi_o$.

Now all the variables involved in the small oscillations are functions of θ, ϕ, ψ, only. Hence, it follows from the above, that initially θ, ψ, f, l, are zero, and in the subsequent motion they remain small quantities of the first order.

Also, for small values of θ and ψ, we have

$$l=-\theta\cos\chi-\psi\sin\chi$$
$$m=-\cos\overline{\phi+\chi}$$
$$n=\sin\overline{\phi+\chi}$$
$$f=h_o(\theta\cos\phi_o-\psi\sin\phi_o)$$
$$h=h_o\frac{\sin\overline{\phi_o+\chi}}{\sin\overline{\phi+\chi}}$$

The conditions of equilibrium are

$$X_o=Tl_o=0$$
$$Y_o=Tm_o-W\cos\phi_o=0$$
$$Z_o=Tn_o+W\sin\phi_o-R_o=0$$
$$L_o=h_oTn_o-\eta_oR_o=0$$
$$N_o=f_om_o-h_ol_o=0.$$

2. *Equations of Motion*:—With the axes taken as above the equations of motion of the kite are

$$M\left(\frac{du}{dt}+w\,\theta_2-v\,\theta_3\right)=X=Tl+W\sin\theta$$
$$M\left(\frac{dv}{dt}+u\,\theta_3-w\,\theta_1\right)=Y=Tm-W\cos\theta\cos\phi$$
$$M\left(\frac{dw}{dt}+v\,\theta_1-u\,\theta_2\right)=Z=Tn+W\cos\theta\sin\phi-R$$
$$A\dot{\theta}_1+(C-B)\theta_2\theta_3=L=hTn-\eta R$$
$$B\dot{\theta}_2+(B-A)\theta_1\theta_3=M=-fTn$$
$$C\dot{\theta}_3+(A-C)\theta_2\theta_1=N=T(fm-hl).$$

The quantities whose variations determine the small oscillations, are the velocity components and the co-ordinates f, h, η. We have already given their values in terms of ϕ, ψ and χ. Hence to find the variations of f, h, η to the first order, we put $\sin\theta=\theta$, $\sin\psi=\psi$, $\phi=\phi_o+\epsilon$ and also assume u, v, w, θ, &c., to be small, so that their squares and products may be neglected.

† Throughout this paper zero suffixes are used to denote the initial values of the quantities involved.

Hence substituting these values and simplifying, we have

$$M\dot{u} = -T(\theta \cos \chi + \psi \sin \chi) + W\theta \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$M\dot{v} = \underline{-T \cos\overline{\phi_0 + \chi} - W \cos\phi_0} + \epsilon(T \sin \overline{\phi_0 + \chi} + W \sin \phi_0) \quad \ldots \quad (2)$$

$$M\dot{w} = \underline{T \sin \overline{\phi_0 + \chi} + W \sin \phi_0 - R_0} + \epsilon\underline{(T \cos \overline{\phi_0 + \chi} + W \cos \phi_0)} - \delta R \quad \ldots \quad \ldots \quad (3)$$

$$A\dot{\theta}_1 = \underline{h_0 \; T \sin \overline{\phi_0 + \chi} - \eta_0 R_0} - R_0 \delta \eta_c - \eta_0 \delta R \quad \ldots \quad \ldots \quad (4)$$

$$B\dot{\theta}_2 = -h_0 \; T\,(\theta \cos \phi - \psi \sin \phi_0) \sin \overline{\phi_0 + \chi} \quad \ldots \quad \ldots \quad (5)$$

$$C\dot{\theta}_3 = h_0 T\,(\theta \sin \phi + \psi \cos \phi) \sin \overline{\phi_0 + \chi} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

The terms underlined disappear, since (as can be easily proved from elementary statics) they are conditions of equilibrium.

3. *Resistance derivatives.* We have next to find δR and $\delta \eta$. The resistance experienced by a lamina moving through air, is a function of the velocity of the air, and also of the inclination of its plane face (called the "angle of attack") to the horizon, in steady motion or equilibrium. Hence if V be the velocity of air and α the angle of attack, we may write

$$R = R\,(V \cos \alpha,\ V \sin \alpha) = R\ (v_0, w_0) \text{ say,}$$

the functional notation R on the right hand side being at present arbitrary. When the velocity components $u, v, \ldots \theta_3$ are impressed, we have

$$R_0 + \delta R = R(u,\ v_0 + v,\ w_0 + w,\ \theta_1,\ \theta_2,\ \theta_3)$$

$$\therefore \qquad \delta R = uR_u + vR_v + wR_w + \theta_1 R_1 + \theta_2 R_2 + \theta_3 R_3.$$

If the kite is strictly a plane surface with an axis of symmetry, the value of R cannot be affected by any reversal of u, θ_2, θ_3;

hence $$R_u = R_2 = R_3 = 0$$

where R_u, R_1 etc., denote the values of $\dfrac{\partial R}{\partial u}, \dfrac{\partial R}{\partial \theta_1}$ etc., in equilibrium.

Thus $$\delta R = vR_v + wR_w + \theta_1 R_1 \quad \ldots \quad \ldots \quad (1)$$

The values of R_v, R_w, R_1, will of course depend on the actual law of resistance assumed, and it is usual to assume that the resistance is proportional to the square of the velocity and also a function of the angle of attack.

Let us assume then

$$R = KSV^2 f(\alpha)$$

K and S being constants and when v, w, θ_1 are impressed the component velocities of any element S, whose distance from the x-axis is y, become $v_0 + v$ and $w_0 + w + y\theta_1$.

$$\therefore\ R_0+\delta R=KS[V^2+2vv_o+2w_0(w+y\Theta_1)][f(\alpha)+\delta\alpha f'(\alpha)],$$

also $\tan\alpha=\frac{w_o}{v}$. $\therefore\ \sec^2\alpha\,\delta\alpha=\frac{v_o\delta w_0-w_0\delta v_0}{v_0^{\,2}}$.

Now remembering that $v_0=V\cos\alpha$, $w_0=V\sin\alpha$, $\delta w_0=w+y\Theta_1$, $\delta v_0=v$, we have

$$\delta\alpha=\frac{(w+y\Theta_1)\cos\alpha-v\sin\alpha}{V};$$

and on equating coefficients with (1)

$$R_v=KSV\ [2f(\alpha)\cos\alpha-f'(\alpha)\sin\ \alpha]\ldots \qquad \ldots \qquad \ldots \quad (i)$$

$$R_w=KSV\ [2f(\alpha)\sin\alpha+f'(\alpha)\cos\alpha] \qquad \ldots \qquad \ldots \quad (ii)$$

$$R_1=KSV\ [2f(\alpha)\sin\alpha+f'(\alpha)\cos\alpha]\ \overline{y} \qquad \ldots \qquad \ldots \quad (iii)$$

where $\overline{y}\ =\Sigma(yS)$.

To find the change in ηR. We assume

$$\eta=c'\phi(\alpha)$$

where c is a quantity depending on the linear dimensions of the kite. We have also

$$\eta R=\int\eta dR,$$

where η is the C. P. of an element dS, the thrust on which is dR or $KV^2f(\alpha)dS$.

When additional velocity components are impressed, α becomes $\alpha+\delta\alpha$ and V^2 becomes

$$(V\cos\alpha+v)^2+(V\sin\alpha+w+y\Theta_1)^2;$$

hence neglecting squares and products of small quantities

$$\eta_0\delta R+R_0\delta\eta=\eta_0(vR_v+wR_w+\Theta_1R_1)$$
$$+\frac{R_0c'\phi'(\alpha)}{V}\ (w\cos\alpha-v\sin\alpha+\overline{y}\Theta_1\cos\alpha).$$

We have also from Euler's geometrical equations, since Θ is small,

$$\Theta_1=\dot{\phi}=\dot{\epsilon}$$

When these values are substituted we shall have six linear differential equations of the first order to determine the motion completely.

4. It will now be noticed that the equations (1)...(6) of the previous article are separable into two groups, the system (2), (3), (4) determine the oscillations of the kite in the $y\,z$ plane, while (1), (5), (6) determine the sideways rotations and translations.

Equations of symmetric oscillations: When the above values of δR and $\delta\eta$ are substituted the system (2), (3), (4) become

$$M\dot{v}=\epsilon(T\sin\phi_o+X+W\sin\phi_o)$$
$$M\dot{w}=-vR_v-wR_w-\Theta_1R_1$$
$$A\dot{\Theta}_1=-\eta_o(vR_v+wR_w+\epsilon_1R_1)$$
$$-\frac{R_o c'\phi'(\alpha)}{V}(w\cos\alpha-v\sin\alpha+\overline{y}\,\Theta_1\cos\alpha)$$

and to solve this system we assume as usual v, w, ϵ proportional to $e^{\lambda t}$ and eliminate the variables v, w, ϵ. The determinantal equation in λ is reduced to

$$a\lambda^4+b\lambda^3+c\lambda^2+d\lambda+e=0\,;$$

since
$$\mathrm{T}\sin\overline{\phi_0+\chi}+\mathrm{W}\sin\phi_0=\mathrm{R}_0$$

$$a=\mathrm{AM}^2$$

$$b=\mathrm{M}^2\left(\mathrm{R}_0\frac{c'\phi'(\alpha)}{\mathrm{V}}\dot{\bar{y}}\cos\alpha+\eta_0\mathrm{R}_1\right)+\mathrm{AMR}_w$$

$$c=\mathrm{M}\frac{c'\phi'(\alpha)}{\mathrm{V}}\mathrm{R}_0\cos\alpha\,(\bar{y}\mathrm{R}_w-\mathrm{R}_1)$$

$$d=\mathrm{M}\left(\eta_0\mathrm{R}_0\mathrm{R}_v-\mathrm{R}_0{}^2\frac{c'\phi'(\alpha)}{\mathrm{V}}\sin\alpha\right)$$

$$e=-\frac{\mathrm{R}_0{}^2}{\mathrm{V}}c'\phi'(\alpha)(\mathrm{R}_w\sin\dot{\alpha}+\mathrm{R}_v\cos\alpha).$$

Asymmetric Oscillations: Here we have the three equations.

$$\mathrm{M}\dot{u}=\mathrm{T}l+\mathrm{W}\theta=\theta(\mathrm{W}-\mathrm{T}\cos\chi)-\psi.\mathrm{T}\sin\chi$$

$$\mathrm{B}\dot{\theta}_2=-f\mathrm{T}n=-h_0\mathrm{T}\sin\overline{\phi_0+\chi}\,(\theta\cos\phi_0-\psi\sin\phi_0)$$

$$\mathrm{C}\dot{\theta}_3=\mathrm{T}(fm-hl)=-h_0\mathrm{T}\cos\overline{\phi_0+\chi}\,(\theta\cos\phi_0-\psi\sin\phi_0)\,;$$

we have also from Euler's geometrical equations, since θ is small,

$$\dot{\theta}=\theta_2\sin\phi_0+\theta_3\cos\phi_0$$

$$\dot{\psi}=\theta_2\cos\phi_0-\theta_3\sin\phi_0.$$

If we differentiate the above three equations, and substitute these values of $\dot{\theta}$ and $\dot{\psi}$, we get on slight simplification

$$\mathrm{M}\ddot{u}=\theta_2(\mathrm{W}\sin\phi_0-\mathrm{T}\sin\overline{\phi_0+\chi})+\theta_3(\mathrm{W}\cos\phi-\mathrm{T}\cos\overline{\phi_0+\chi})$$

$$=p_2\theta_2+p_3\theta_3 \text{ say}$$

$$\mathrm{B}\ddot{\theta}_2=h_0\mathrm{T}\sin\overline{\phi_0+\chi}\cdot\theta_3$$

$$\mathrm{C}\ddot{\theta}_3=h_0\mathrm{T}\sin\overline{\phi_0+\chi}.\theta_2\,;$$

and to solve this system, we assume as before u, θ_2, θ_3, proportional to $e^{\lambda t}$, and the determinantal equation in λ is

$$\begin{vmatrix} \mathrm{M}\lambda^2 & -p_2 & -p_3 \\ 0 & \mathrm{B}\lambda^2 & h_0\mathrm{T}\sin\overline{\phi_0+\chi} \\ 0 & h_0\mathrm{T}\sin\overline{\phi_0+\chi} & -\mathrm{C}\lambda^2 \end{vmatrix}=0,$$

i.e.
$$\lambda^2(\mathrm{BC}\lambda^4+h_0{}^2\,\mathrm{T}^2\sin^2\overline{\phi_0+\chi})=0.$$

5. *Conditions of Stability*. The dependence of stability on the roots of these equations will be found fully discussed in Routh, vol. II, chap. vi. For stability, the real roots or the real part of the imaginary roots

must be negative, and this will be the case, all the coefficients being finite, if a, b, c, d, e and $(bcd-eb^2-ad^2)$ have the same sign.

It follows, since a is positive, that all the coefficients should be positive.

First of all, we notice that if in the equations for symmetric oscillations, we neglect the displacement of the centre of pressure, and the position of equilibrium be such that the string passes through the centre of gravity, then all the coefficients in the biquadratic with the exception of the first two vanish, indicating lack of longitudinal stability. This result has an important application to aeroplanes.†

If the displacement of the centre of pressure be not neglected, then we have, since $R_w \sin \alpha + R_v \cos \alpha = 2\ KSVf(\alpha)$,

$$e = -\frac{R_o^2}{V} c\phi'(\alpha) . 2KSVf(\alpha)$$

which shows that for stability $\phi'(\alpha)$ must be negative, or the centre of pressure must move forward as the angle of attack diminishes. This condition is satisfied in practice.

With regard to lateral stability, we have obtained the equations on the assumption that the resultant tension does not intersect the axis of symmetry. But if the string simply bifurcates into two branches, then it can be easily proved from considerations of elementary statics that the string always intersects the axis of symmetry.

Let H be the point of bifurcation and E the point where the string meets the ground. Let the two points of attachment to the kite be C and D. Now if the direction of the string be fixed in space, then H is a fixed point, and the three forces acting on it are the three tensions HC, HD, and HE; and for the equilibrium of H these must lie in one plane, or EH produced must intersect CD.

In this case $M=0$ and $N=-hTl$; and it can be easily proved in the same manner as before that the coresponding equation in λ is

$$M\lambda^2\left(\lambda^2 + \frac{h_0 W_c \sin \alpha}{C}\right) = 0,$$

a biquadratic with two zero roots.

We conclude, in any case, that in strictly plane kites lateral stability is impossible so long as the centre of pressure does not deviate

† See : *Stability in Aviation* by G. H. Bryan, F. R. S. (Macmillan)

from the axis of symmetry‡. There are several other facts which yet remain to be explained, namely the peculiar behaviour of the kite when the wind velocity changes and the necessity of adjusting the two lengths HC and HD properly. To discuss these cases it will be necessary to take into account the variation in the magnitude and the direction of the tension. The next paper will be devoted to the discussion of the case of longitudinal stability with finite string, and explanation of some well-known facts connected with kite-flying.

I have to thank Prof. Bryan and Mr. Berwick for many valuable suggestions.

(*To be continued*).

‡ In practice the lateral stability is increased by the bending of the plane of the kite under wind pressure.

SHORT NOTES.

The Mathematical Association of America.

On December 30 and 31, 1915, there was held at Columbus, Ohio, the organisation meeting of a new national mathematical association, the call for which had been signed by 450 persons representing every state in the Union, the District of Columbia, and Canada. The object of the new Association is to assist in promoting the interests of mathematics in America, especially in the collegiate field. It is not intended to be a rival of any existing organisation, but rather to supplement the Secondary Associations on the one hand, and the American Mathematical Society on the other, the former being well organized and effective in their field, and the latter having definitely limited itself to the field of scientific research. In the field of collegiate mathematics, however, there has been, up to this time, no organization and no medium of communication among the teachers, except the American Mathematical Monthly, which for the past three years has been devoted to this cause. The new organization, which has been named the Mathematical Association of America, has taken over the Monthly as its Official Journal.

There were 104 persons present at the organization meeting. The constitution and by-laws together with a full report of the proceedings will be published in the January issue of the Monthly. The following officers were elected :

President, Professor E. B. Hedrick, University of Missouri.

First Vice-President, Professor E. V. Huntington, Harvard University.

Second Vice-President, Professor G. A. Miller, University of Illinois.

Secretary-Treasurer, Professor W. D. Cairns, Oberlin College.

Publication Committee, Professor H. E. Slaught, University of Chicago, Managing Editor, Professor W. H. Bussey, University of Minnesota, and Professor R. D. Carmichael, University of Illinois.

These officers, together with the following, constitute the Executive Council :

Professor R. C. Archibald, Brown University.

Do. Florian Cajori, Colorado College.

Do. B. F. Finkel, Drury College.

Do. D. N. Lehmer, University of California.

Professor	E. H. Moore, University of Chicago.
Do.	R. E. Moritz, University of Washington.
Do.	M. B. Porter, University of Texas.
Do.	K. D. Swartzel, Ohio State University.
Do.	J. N. Van der Vries, University of Kansas.
Do.	Oswald Veblen, Princeton University.
Do.	J. W. Young, Dartmouth College.
Do.	Alexander Ziwet, University of Wisconsin.

Note on Partial Fractions.

1. *Lemma.*—Suppose λ is a root of the equation

$$\phi(x) \equiv a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_n = 0\,;$$

then any fraction $f(\lambda)/F(\lambda)$ can be reduced to an integral expression of the $(n-1)^{th}$ degree in λ.

This is known as Tschirnhaussen's transformation and is explained in books on *Theory of Equations*. The following method of proof may be of interest.

By the method of continued fractions the fraction $\phi(\lambda)/F(\lambda)$ can be written in the form

$$A_0 + \frac{1}{A_1 +}\,\frac{1}{A_2 +}\cdots$$

where the A's are functions of λ. If the penultimate convergent to the continued fraction be P/Q, we have

$$P.F(\lambda) - Q\cdot\phi(\lambda) = \pm 1\,;$$

so that $\quad P.F(\lambda) = \pm 1$, or $f(\lambda)/F(\lambda) = \pm f(\lambda).P$,

which is an integral function. Evidently, this integral function can be written

$$\phi(\lambda).Q' + R$$

by division by $\phi(\lambda)$, where R is of the $(n-1)$th degree in λ.

Hence, the given fraction

$$f(\lambda)/F(\lambda) = \phi(\lambda)\cdot Q' + R = R,$$

which proves the Lemma.

2. To reduce any expression

$$\frac{\psi(x)}{\Pi\,[\phi(x)]}$$

where the denominator consists of different real integral factors $\phi\,(x)$, we proceed thus:

Put $$\frac{\psi(x)}{\Pi\,[\phi(x)]} = \Sigma\left[\frac{\phi_1\,(x)}{\phi\,(x)}\right]; \quad \ldots \quad \ldots \quad (1)$$

and multiply both sides by $\phi\,(x)$ and afterwards write λ for x, where λ is a root of $\phi\,(x) = 0$.

Then by the foregoing Lemma, the left hand side can be reduced to an integral function R of a degree lower than $\phi(x)$; and the right hand side will be $\phi_1(\lambda)$.

Hence $$R=\phi_1(\lambda);$$

In other words, *the numerator* $\phi_1(x)$ *of the partial fraction corresponding to* $\phi(x)$ *is the transform of the given fraction after multiplication by* $\phi(x)$.

3. As an example consider the case of the cubic

$$\phi(x)\equiv x^3+px^2+qx+r,$$

and the corresponding numerator Px^2+Qx+R.

Write $f(x)$ for the left hand member of (1) multiplied by $\phi(x)$. Then the values of P, Q, R may also be directly obtained by the following artifice.

Let α, β, γ be the roots of $x^3+px^2+qx+r=0$, so that

$$P\alpha^2+Q\alpha+R-f(\alpha)=0,$$
$$P\beta^2+Q\beta+R-f(\beta)=0,$$
$$P\gamma^2+Q\gamma+R-f(\gamma)=0;$$

solving these for P, Q, R, we get

$$\frac{P}{\begin{vmatrix} 1 & \alpha & f(\alpha) \\ 1 & \beta & f(\beta) \\ 1 & \gamma & f(\gamma) \end{vmatrix}}=\frac{Q}{\begin{vmatrix} 1 & \alpha^2 & f(\alpha) \\ 1 & \beta^2 & f(\beta) \\ 1 & \gamma^2 & f(\gamma) \end{vmatrix}}=\frac{R}{\begin{vmatrix} \alpha & \alpha^2 & f(\alpha) \\ \beta & \beta^2 & f(\beta) \\ \gamma & \gamma^2 & f(\gamma) \end{vmatrix}}=\frac{-1}{\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}}.$$

The first three determinants are divisible by the last, and the quotients being symmetric functions of α, β, γ are readily expressed in terms of p, q, r. Any of the first three determinants can be expanded by Maclaurin's theorem and the result expressed in a series thus:

$$\begin{vmatrix} 1 & \alpha & f(\alpha) \\ . & . & . \\ . & . & . \end{vmatrix}=\Sigma(D_r),$$

where $D_r=\dfrac{f^r(o)}{r!}\begin{vmatrix} 1 & \alpha & \alpha^r \\ 1 & \beta & \beta^r \\ 1 & \gamma & \gamma^r \end{vmatrix}=S_{r-2}.\dfrac{f^r(o)}{r!}\times\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}$

Hence $$P=-\Sigma\left[\frac{f^r(o)}{r!}S_{r-2}\right]$$

where S_{r-2} = sum of homogeneous products of α, β, γ of $(r-2)$ dimensions.

M. T. NARANIENGAR.

Newton's Theorems.

1. Let $S_r \equiv a\alpha^r + b\,\beta^r + c\,\gamma^r + \ldots$

where $\alpha, \beta, \gamma \ldots$ are all the roots of the equation

$$f(x) \equiv x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots\ldots (-1)^n p_n = 0, \quad \ldots \quad \ldots \quad (1)$$

and a, b, c are any given numbers.

Multiply (1) by x^{m-n}, and substitute $\alpha, \beta, \gamma \ldots$ for x in succession; multiply the results by $a, b, c \ldots$ in order and add. We have

$$s_m - p_1 s_{m-1} + p_2 s_{m-2} \ldots\ldots (-1^n \cdot s_{m-n} = 0 \; [m > n] \quad \ldots \quad \ldots \quad (2)$$

which is an extension of Newton's first theorem.

2. When $m < n$, we find

$$s_m - p_1 s_{m-1} + p_2 s_{m-2} \ldots - (-1)^m (p_m s_0 - q_m) = 0, \quad \ldots \quad (3)$$

where q_m denotes the sum of products of the roots with coefficients equal to the sums of $a, b, c \ldots$ corresponding to the roots not occurring in the product.

For, by the above process,

$$s_{n-1} - p_1 s_{n-2} + p_2 s_{n-3} - \ldots\ldots (-1)^n p_n s_{-1} = 0.$$

But $$p_n s_{-1} = \alpha\beta\gamma \ldots (a\alpha^{-1} + b\beta^{-1} + c\gamma^{-1} + \ldots)$$
$$= a\beta\gamma\delta \ldots + b\alpha\gamma \ldots + \ldots = q_{n-1}.$$

Similarly

$$s_{n-2} - p_1 s_{n-3} + \ldots (-1)^{n-2} \{ p_{n-2} s_0 - p_{n-1} s_{-1} + p_n s_{-2} \} = 0,$$

and

$$p_{n-1} s_{-1} - p_n s_{-2} = \alpha\beta\gamma \ldots \left\{ \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \ldots \right) \left(\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} + \ldots \right) - \left(\frac{a}{\alpha^2} + \frac{b}{\beta^2} + \frac{c}{\gamma^2} + \ldots \right) \right\}$$

$$= \alpha\beta\gamma \ldots \left(\frac{a+b}{\alpha\beta} + \frac{b+c}{\beta\gamma} + \ldots \right)$$

$$= q_{n-2};$$

and so on.

3. Again, suppose

$$\phi(\alpha) = a_0 \alpha^m + a_1 \alpha^{m-1} + \ldots + a_m;$$

then writing

$$s_r = \phi^r(\alpha) + \phi^r(\beta) + \phi^r(\gamma) + \ldots,$$

we obtain

$$q_0 s_m + q_1 s_{m-1} + \ldots m\, q_m = 0, \; [m < n] \quad \ldots \quad (4)$$
$$q_0 s_m + q_1 s_{m-1} + \ldots q_n s_{m-n} = 0, \; [m > n] \quad \ldots \quad (5)$$

where the q's are the co-efficients of the equation whose roots are $\phi(\alpha)$, $\phi(\beta)$, $\phi(\gamma)$, etc.

4. *Examples*:

(i) Let $s_1 = \Sigma(\alpha^p)$, $s_2 = \Sigma(\alpha^{2p})$, ..., then $q_0, q_1 \ldots$ are the co-efficients of the rationalised equation $f\left(y^{1/p}\right) = 0$.

(ii) Suppose $f(x)=(x-\alpha)^a.(x-\beta)^b.(x-\gamma)^c....$
$$=x^n+P_1x^{n-1}+P_2x^{n-2}+...P_n.$$

Then $\quad f'(x)/f(x)=a/(x-\alpha)+b/(x-\beta)+c/(x-\gamma)+...$
$$=\Sigma\{s_r/x^{r+1}\},$$

where $s_r=a\alpha^r+b\beta^r+c\gamma^r+.. ...$

Proceeding as in Cajori's *Theory of Equations*, § 68, we get

$$\left.\begin{array}{l} s_1+n\ P_1=(n-1)\ P_1......... \\ s_2+P_1\ s_1+n\cdot P_2=(n-2)P_2 \\ ...\quad ...\quad ... \end{array}\right\} \quad \therefore \quad \begin{array}{l} s_1+P_1=0, \\ s_2+P_1\ s_1+2P_2=0, \\ ...\quad ...\quad ... \end{array}$$

where

$n=a+b+c...\ ;-P_1=a\alpha+b\beta+c\gamma+...=s_1\ ;$

$P_2=$sum of the products of roots two at a time

$$=ab\alpha\beta+bc\beta\gamma+...+\frac{a(a-1)}{1.2}\alpha^2+\frac{b(b-1)}{1\ 2}\beta^2+\frac{c(c-1)}{1.2}\gamma^2+...$$

$$=\tfrac{1}{2}\{(a\alpha+b\beta+...)^2-(a\alpha^2+b\beta^2+...)\}=\tfrac{1}{2}(s_1^2-s_2):\ \&c.,\ \&c.$$

Hence also we can derive the result (3) above.

(iii) To find the identical relations satisfied by s_1, s_2..., we write down $(n+1)$ relations for particular values of m in (2) or (3) and eliminate the p's. Thus, in the case of the cubic

$$x^3-p_1x^2+p_2x-p_3=0,$$

we may take

$$s_3-p_1s_2+p_2s_1-p_3s_0=0$$
$$s_4-p_1s_3+p_2s_2-p_3s_1=0$$
$$s_5-p_1s_4+p_2s_3-p_3s_2=0$$
$$s_6-p_1s_5+p_2s_4-p_3s_3=0$$

and eliminate p_1, p_2, p_3 to obtain the identical relation

$$\begin{vmatrix} s_3 & s_2 & s_1 & s_0 \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ ... & ... & ... & ... \end{vmatrix}=0.$$

M. T. NARANIENGAR.

The Face of the Sky for May and June

The Sun

enters Gemini on May 21 and the Summer solstice on June 21 at 11-54 P.M.

The Moon

	May.			June.		
	D.	H.	M.	D.	H.	M.
New Moon ...	2	10	59	1	1	7
First Quarter ...	10	2	17	9	5	29
Full Moon ...	17	7	41	16	3	12
Last Quarter ...	24	10	46	22	6	46
New Moon ...		...		... 30	4	13

The Planets.

Mercury attains its greatest elongation (E) on May 12. It is stationary on May 25 and is in inferior conjunction on June 6. It is stationary on June 18 and attains its greatest elongation 21° 50′ (W) on June 30. It is in conjunction with the Moon on May 4, June 1 and June 28.

Venus continues an evening star. It attains its greatest brilliancy on June 1. It is stationary on June 12. It is in conjunction with the Moon on May 6 and on June 4 and with Saturn on May 24 and June 22.

Mars is in quadrature to the Sun on May 15. It is in conjunction with the Moon on May 10 and June 8.

Jupiter is in conjunction with the Moon on May 28, and on June 25 at 5-11 A. M.

Saturn is in conjunction with the Moon on May 7 and June 3 at 11-48 P.M. and with 8 Geminorum on June 4.

Uranus is in quadrature to the Sun on May 10. It is stationary on May 25. It is in conjunction with the Moon on May 23 and June 19.

Neptune is in conjunction with the Moon on May 8 and June 5.

V. RAMESAM.

SOLUTIONS.

Questions 586 & 628.

(K. V. ANANTANARAYANA SASTRI, B.A.) :—Prove that the length of the first negative pedal of the loop of the Folium of Descartes $x^3+y^3-3\,axy=0$, is $6a-a\,[\pi-\sqrt{2}\ \ \log(1+\sqrt{2})]$.

Solution by T. P. Krishnaswami.

It is known that the negative pedal of any curve is the envelope of a line perpendicular to the radius rector through its other extremity. In the present case the line is

$$x\cos\alpha+y\sin\alpha=\frac{3\,at\,\sqrt{1+t^2}}{1+t^3}\text{, where } t=\tan\alpha$$

$$\text{i.e.}\quad x\cos\alpha+y\sin\alpha=\frac{3\,a\sin\alpha\cos\alpha}{\sin^3\alpha+\cos^3\alpha}\qquad\ldots\qquad\ldots\qquad\ldots\qquad(1)$$

Again, it is also known that the radius of curvature of the envelope of of the line

$$x\cos\alpha+y\sin\alpha=f(\alpha)\qquad\ldots\qquad\ldots\qquad\ldots\qquad(2)$$

is $f(\alpha)+f''(\alpha)$. : (See Edward's *Diff. Calc.*, P. 309 Ex. 17) and, as α and ψ differ only by a constant quantity, $d\alpha=d\psi$ and therefore we get from (2) that the arc

$$s=\int[f(\alpha)+f''(\alpha)]\,d\alpha$$

$$=\int_{\frac{1}{2}\pi}^{0}\frac{3a\sin\alpha\cos\alpha}{\sin^3\alpha+\cos^3\alpha}\,d\alpha+\left[\frac{d}{d\alpha}\left(\frac{3a\sin\alpha\cos\alpha}{\sin^3\alpha+\cos^3\alpha}\right)\right]_{\frac{1}{2}\pi}^{0}$$

$$=\int_{\frac{1}{2}\pi}^{0}\frac{3a\sin\alpha\cos\alpha}{\sin^3\alpha+\cos^3\alpha}\,d\alpha+6a.$$

Now, since $y=x$ is a line of symmetry, the integral on the right side can be written

$$2a\int_{0}^{\frac{1}{4}\pi}\left\{\frac{1}{\sin\alpha+\cos\alpha}-\frac{\sin\alpha+\cos\alpha}{1-\sin\alpha\cos\alpha}\right\}d\alpha$$

$$=a\sqrt{2}\int_{0}^{\frac{\pi}{4}}\frac{d\alpha}{\cos\left(\alpha-\frac{\pi}{4}\right)}\cdot 4-a\sqrt{2}\int_{0}^{\frac{1}{4}\pi}\frac{\cos\left(\alpha-\frac{\pi}{4}\right).}{2-\cos 2\left(\alpha-\frac{\pi}{4}\right)}$$

$$=a\sqrt{2}\left[\log\tan\left(\frac{\pi}{4}+\frac{\pi}{2}-\frac{\pi}{8}\right)\right]_{0}^{\frac{1}{4}\pi}$$

$$-4a\sqrt{2}\int_0^{\frac{\pi}{4}} \frac{d\sin\left(\alpha-\frac{\pi}{4}\right)}{1+2\sin^2\left(\alpha-\frac{\pi}{4}\right)}$$

$$=-a\sqrt{2}\log\left(\tan\frac{\pi}{8}\right)-4a\tan^{-1}\left\{\sqrt{2}\sin\left(\alpha-\frac{\pi}{4}\right)\right\}$$

$$=a\sqrt{2}\log(\sqrt{2}+1)-a\pi.$$

Hence $s=6a-a\pi+a\sqrt{2}\log(\sqrt{2}+1)$.

Question 630.

(S. NARAYANA AIYAR, M. A) :—If

$$A_n=\frac{(a-b)(a-bx)\ldots\ldots\ldots(a-bx^{n-1})}{(1-x)(1-x^2)\ldots\ldots\ldots(1-x^n)},$$

and

$$B_n=\frac{(b-a)(b-ax)\ldots\ldots\ldots(b-ax^{n-1})}{(1-x)(1-x^2)\ldots\ldots\ldots(1-x^n)},$$

shew that

$$(-1)^n B_n=\begin{vmatrix} A_1, & 1, & 0, & 0 & \ldots & \ldots & \ldots & 0 \\ A_2, & A_1, & 1, & 0 & \ldots & \ldots & \ldots & 0 \\ A_3, & A_2, & A_1, & 1 & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n-1}, & A_{n-2}, & A_{n-3}, & A_{n-4} & \ldots & \ldots & \ldots & 1 \\ A_n, & A_{n-1}, & A_{n-2} & A_{n-3}, & \ldots & \ldots & \ldots & A_1 \end{vmatrix}$$

with a similar determinant for $(-1)^n A_n$ in terms of the constituents $B_1, B_2, \ldots\ldots B_n$.

Solution by N. Sankara Aiyar, M. A.

The given determinant is equal to

$$A_1\begin{vmatrix} A_1 & 1 & 0 & \ldots & 0 \\ A_2 & A_1 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n-1} & A_{n-2} & \ldots & \ldots & A_1 \end{vmatrix} - \begin{vmatrix} A_2 & 1 & 0 & \ldots & 0 \\ A_3 & A_1 & 1 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & & \\ A_n & A_{n-2} & \ldots & \ldots & A_1 \end{vmatrix}$$

We shall prove the result by Mathematical Induction.

We should prove that

$$\begin{vmatrix} A_2 & 1 & 0 & \dots & 0 \\ A_3 & A_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A_n & A_{n-2} & \dots & \dots & \dots \end{vmatrix} = A_1(-1)^{n-1}B_{n-1} - (-1)^n B_n$$

$$= (-1)^{n-1}B_{n-1}\left(\frac{a-b}{1-x} + \frac{b-ax^{n-1}}{1-x^n}\right)$$

$$= (-1)^{n-1}B_{n-1}\frac{(a-bx)(1-x^{n-1})}{(1-x)(1-x^n)};$$

that is,

$$(-1)^{n-2}B_{n-2}\cdot A_1 - \begin{vmatrix} A_3 & 1 & \dots & 0 \\ A_4 & A_1 & \dots & 0 \\ A_n & A_{n-3} & \dots & A_1 \end{vmatrix} = (-1)^{n-1}B_{n-1}\frac{a-bx}{(1-x)}\frac{1-x^{n-1}}{1-x^n}$$

$$\therefore \begin{vmatrix} A_3 & 1 & \dots & \dots & 0 \\ A_4 & A_1 & \dots & \dots & 0 \\ \dots & \dots & \dots & & \\ A_n & A_{n-3} & \dots & A_1 & \end{vmatrix} = (-1)^{n-2}B_{n-2}\frac{a-bx}{1-x}\left(\frac{a-b}{1-x^2} + \frac{b-ax^{n-2}}{1-x^n}\right)$$

$$= (-1)^{n-2}B_{n-2}\cdot\frac{a-bx}{1-x}\cdot\frac{a-bx^2}{(1-x^2)}\cdot\frac{1-x^{n-2}}{1-x^n};$$

and so on, until, finally, we have

$$\begin{vmatrix} A_{n-1} & 1 \\ A_n & A_1 \end{vmatrix} = (-1)^2 B_2 \frac{a-bx}{1-x}\cdots\frac{a-bx^{n-2}}{1-x^{n-2}}\cdot\frac{1-x^2}{1-x^n}.$$

But $A_{n-1}A_1 - A_n = A_{n-1}\left(\frac{a-b}{1-x} - \frac{a-bx^{n-1}}{1-x^n}.\right)$

$$= -A_{n-1}\frac{b-ax}{1-x}\cdot\frac{1-x^{n-1}}{1-x^n}.$$

$$= \frac{b-ax}{1-x}\cdot\frac{b-a}{1-x}\cdot\frac{a-bx}{1-x^2}\cdots\frac{a-bx^{n-2}}{1-x^{n-1}}\cdot\frac{1-x^{n-1}}{1-x^n}.$$

Hence we should get

$$\frac{b-a}{1-x}\cdot\frac{b-ax}{1-x^2}\cdot\frac{a-bx}{1-x}\cdots\frac{a-bx^{n-2}}{1-x^{n-1}}\cdot\frac{1-x^2}{1-x^n}$$

$$= \frac{b-a}{1-x}\cdot\frac{b-ax}{1-x}\cdot\frac{a-bx}{1-x^2}\cdots\frac{a-bx^{n-2}}{1-x^{n-1}}\cdot\frac{1-x^{n-1}}{1-x^n},$$

which is obvious.

Thus, if the formula is true up to $n-1$, it is true up to n. Hence, &c. Similarly the case of A_n in terms of the B's can be proved.

Question 639.

(V. V. S. NARAYAN):—The polar circles of the five triangles[2] external to a pentagon, which are formed by producing its sides have a common orthogonal circle.

Solution by K. B. Madhava.

Let P be the conic touching the sides of the pentagon A B C D E, and Q its director circle. If the $\triangle$ external to the pentagon formed by producing the sides A E & B C be called $\triangle_1$; and the polar circle of this triangle R_1, and so on; then, by Gaskin's Theorem (Durell, Theorem 215; or Salmon, § 375, Ex 2)

R_1 is orthogonal to Q.

Similarly all the circles $R_2 \ldots R_5$ are orthogonal to Q. Hence the result.

Question 641.

(SELECTED): – " There cannot be five prime numbers in arithmetical progression unless their common difference be divisible by $2\times3\times5=30$ except when the first term of the progression is 5 ".

[Barlow's *Theory of Numbers*, p, 65.]

Illustrate or criticise the above statement.

Solution by N. Sankara Aiyar, M.A., and R. D. Karve.

Let the five prime numbers be

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d.$$

First, a and d are prime to each other and $a \not\equiv 3$; for otherwise $a+3d$ would not be prime; a may be 5 and then $d \not\equiv 0$ (mod 5).

If a is greater than 5 then one of the four numbers

$$a+d,\ a+2d,\ a+3d,\ a+4d$$

will be $\equiv 0$ (mod 5), unless $d\equiv0$(mod 5).

Similarly, one of $a+d$, $a+2d$, $a+3d$ will be $\equiv 0$ (mod 3), unless $d\equiv0$ (mod 3); and so also $d\equiv0$ (mod 2).

Hence $d\equiv0$ (mod 30), unless $a=5$, in which case $d\equiv0$ (mod 6).

Examples:—(1) 5, 11, 17, 23, 29;
(2) 5, 17, 29, 41, 53;
(3) 5, 47, 89, 131, 173;
(4) 5, 53, 101, 149, 197;
(5) 71, 101, 131, 161, 191;
(6) 71, 131, 191, 251, 311;
(7) 73, 223, 373, 523, 673;
(8) 89, 599, 1109, 1619, 2129.

It must however be noted that if $d \neq 5$, there is the possibility of a sixth number of the A.P. being prime. Thus if $a=7$, $d \neq 0 \pmod 7$ $a \neq 7$ we have

$$7, 37, 67, 97, 127, 157;$$
$$7, 157, 307, 457, 607, 757;$$
$$11, 71, 131, 191, 251, 311;$$
$$11, 491, 971, 1451, 1931, 2411.$$

But there will not be seven such numbers, unless either $a=7$ or $d=0 \pmod{210}$. Thus in the second set above 907 also is prime.

Question 649.

(S. Malhari Rao, B.A.) :—A gentleman has one third of an acre of vacant land round his house. He wishes to divide it into five different plots to grow five different kinds of flowers. What must be the areas of these plots so that they may be in A. P., and each of them may be contained an exact number of times in an acre? Shew that there cannot be more than one set of values for the areas.

Solution by V. Anantaraman and H. K. Chakrabarty, B.Sc.

Let the unit area be an acre. Then, since the area of the five plots is $\frac{1}{3}$ and they are in A. P., the area of the middle plot should be $\frac{1}{15}$.

Suppose the second and fourth plots are $\frac{1}{x}, \frac{1}{y}$; then by the question

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{15},$$

i.e.
$$15(x+y)=2xy,$$

where x and y are positive integers.

Now xy must be divisible by 15, and the following two distinct cases arise:

(i) $x=15a$; so that $15a+y=2ay$, whence $y=ab$ and $15+b=2ab$, resulting in

$$b=1,\ a=8,\ y=8,\ x=120$$
$$b=3,\ a=3,\ y=9,\ x=45$$
$$b=5,\ a=2,\ y=10,\ x=30$$
$$b=15,\ a=1,\ y=15,\ x=15.$$

(ii) $\left.\begin{array}{l} x=5a \\ y=3b \end{array}\right\}$, so that $5a+3b=2ab$, whence $5a/b$ is an integer, resulting in $b=5$, $a=3$, $x=y=15$;

or $\quad a=bt$, $5t+3=2bt$ and $t=1, b=4, a=4, x=20, y=12$;

$$t=3,\ b=3,\ a=9,\ x=45,\ y=9.$$

Thus the possible values of x and y are

$$\left.\begin{array}{l} x=120, 45, 30, 20, 15 \\ y=\ 8,\ 9,\ 10,\ 12,\ 15. \end{array}\right\}$$

Also, since the middle plot is $\frac{1}{15}$, twice the common difference is less than $\frac{1}{15}$. That is

$$\frac{1}{y}-\frac{1}{x}<\frac{1}{15}.$$

Hence, the only admissible values of x and y are 20 and 12; and the singular solution is

$$\frac{1}{30}, \frac{1}{20}, \frac{1}{15}, \frac{1}{12}, \frac{1}{10}.$$

Question 651.

(S. MALHARI RAO, B. A.):—The Nasik or pan-diagonal Magic square (*a*) has been constructed by placing the series of natural numbers from 1 up to 25 along the combined paths of a Knight and a Rook in chess. By adopting another method, and guided by the altered positions 1, 7, 13, 19 and 25 as shown in square (*b*), fill up the vacant cells in it so that it may be magic and pan-diagonal.

(*a*)

12	25	8	16	4
18	1	14	22	10
24	7	20	3	11
5	13	21	9	17
6	12	2	15	23

(*b*)

13				
	19			
		25		
			1	
				7

Solution by H. R. Chakrabarty, B. Sc.

The completed square is

13	10	2	2	16
22	19	11	8	5
6	3	25	17	14
20	12	9	1	23
4	21	18	15	7

Here the method adopted is the combined path of a Knight and a Bishop in chess. Knight's path is the main course, but when obstructed, Bishop's path is taken.

Question 652.

(S. P. SINGARAVELU MUDALIAR) :—From a point (eccentric angle ϕ) of an ellipse of semi-axes a, b three normals are drawn to the ellipse ; show that the square of the radius of the circle passing through the feet of the normals is

$$\left(a+\frac{b^2}{2a}\right)^2\cos^2\phi+\left(b+\frac{a^2}{2b}\right)^2\sin^2\phi.$$

Solution (1) *by M. V. Arunachala Sastry, M.A.* (2) *by N. Sankara Aiyar, M.A.*

(1) The circle in question is evidently Joachimsthal's circle of the point ($a\cos\phi$, $b\sin\phi$). Its equation is given in page 219, equation (549), in Casey's Analytical Geometry.

If, in equation (549), we put $x'=a\cos\phi$, $y'=b\sin\phi$, and $u=a^2+b^2$, we get, as the required equation of the circle in question,

$$x^2+y^2-\frac{b^2}{a}x\cos\phi-\frac{a^2}{b}y\sin\phi=a^2+b^2.$$

[See Ex. 43, page 184, of Asquith's *Analytical Geometry.*]

Hence the square on the radius

$$=\left(a+\frac{b^2}{2a}\right)^2\cos\phi+\left(b+\frac{a^2}{2b}\right)\sin^2\phi.$$

(2) The equation of the circle through λ_1, λ_2, λ_3 is

$$x^2+y^2-\frac{a^2-b^2}{4aN}(l+m)x-\frac{a^2-b^2}{4ibN}(l-m)y$$
$$+\frac{(MN+L)(a^2-b^2)-2N(a^2+b^2)}{4N}=0$$

where $l=N^2+M$ and $m=LN+1$. (See. J.I.M.S., Q. 638.)

In the given case the equation giving the three values of λ is easily seen to be

$$(a^2-b^2)\cos\theta\cos\phi-a^2\sin\theta\cos\phi+b^2\cos\theta\sin\phi=0.$$

i.e. $$(a^2-b^2)\lambda^3\mu-\lambda^2(a^2+b^2)-\lambda(a^2+b^2)+a^2-b^2=0,$$

on simplification.

$$\therefore\qquad L=\frac{a^2+b^2}{a^2-b^2}\frac{1}{\mu},\ M=-\frac{a^2+b^2}{a^2-b^2},\ N=-\frac{1}{\mu}.$$

$$\therefore\ L=MN\ ;\ \frac{(MN+L)(a^2-b^2)-2n(a^2+b^2)}{4N}=-(a^2+b^2).$$

$$\therefore\ N^2+M+LN+1=(N^2+1)(M+1),$$
$$N^2+M-LN-1=(N^2-1)(1-M).$$

$$\therefore\ N^2+M+LN+1=-\frac{\mu^2+1}{\mu^2}\ \frac{2b^2}{a^2-b^2}=-\frac{4b^2\cos\phi}{\mu(a^2-b^2)}.$$

$$N^2+M-LN-1=\frac{\mu^2-1}{\mu^2}\ \frac{2a^2}{a^2-b^2}=-\frac{4a^2\sin\phi}{\mu(a^2-b^2)}.$$

The square on the radius

$$=\left\{\frac{a^2-b^2}{8aN}(l+m)\right\}^2+\left\{\frac{a^2+b^2}{8ibN}(l-m)\right\}^2+a^2+b^2$$

$$=\left(\frac{b^2}{2a}\cos\phi\right)^2+\left\{\frac{a^2}{2b}\sin\phi\right\}^2+a^2+b^2$$

$$=a^2+b^2+\frac{b^4}{4a^2}\cos^2\phi+\frac{a^4}{4b^2}\sin^2\phi.$$

$$=\left(a+\frac{b}{2a}\right)^2\cos^2\phi+\left(b+\frac{a}{2b}\right)^2\sin^2\phi.$$

Question 657.

(T. P. Thrivedi, M.A., LL.B.) :—Prove that the value of the determinant of the $(n+2)^{th}$ order

$$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & a_1+a_2 & \dots & a_1+a_{n+1} \\ 1 & a_1+a_2 & 0 & \dots & a_2+a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1+a_{n+1} & a_2+a_{n+1} & \dots & 0 \end{vmatrix}$$

is equal to $-(-2)^n\,\Pi\,a_1\,\Sigma\,1/a_n$). Also find the value of the determinant of the $(n+1)^{th}$ order obtained by deleting the first column and first row.

Solution by H. K. Chakrabarty, K. B. Madhava and A. Narasinga Rao.

Let the determinant $\begin{vmatrix} -1 & 1 & 1 & \dots & 1 \\ 1 & -1 & 1 & \dots & 1 \\ 1 & 1 & -1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & -1 \end{vmatrix}$

of the $(n+1)^{th}$ order be called Δ_{n+1} and the given determinant D_{n+2}, while that obtained by deleting the first column and first row of D_{n+2} is called D'_{n+1}. Then,

$$\Delta_{n+2}=\frac{1}{n-2}\begin{vmatrix} -2(n-1) & 1 & 1 & \dots & 1 \\ 0 & -1 & 1 & \dots & 1 \\ 0 & 1 & -1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 1 & \dots & -1 \end{vmatrix}$$

obtained by subtracting all the other columns from $n-2$ times the first.

That is

$$\begin{aligned} \Delta_{n+1} &= -2\frac{n-1}{n-2}\Delta_n \\ \Delta_n &= -2\frac{n-2}{n-3}\Delta_{n-1} \\ \dots & \quad \dots \quad \dots \\ \Delta_4 &= -2\frac{2}{1}\Delta_3 \\ \Delta_3 &= (-2)^2. \end{aligned}$$

Hence multiplying by columns

$$\Delta_{n+1}=(-2)^n(n-1).$$

Now $$D_{n+2}=\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & -a_1 & a_2 & \dots & a_{n+1} \\ 1 & a_1 & -a_2 & \dots & a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1 & a_2 & \dots & -a_{n+1} \end{vmatrix}$$

obtained by multiplying the 1st row of D_{n+2} by a_1, a_2 etc., and subtracting from the 2nd, 3rd etc., row respectively.

$$=\Pi a_1\begin{vmatrix} 0 & \frac{1}{a_1} & \frac{1}{a_2} & \dots & \frac{1}{a_{n+1}} \\ 1 & -1 & 1 & \dots & 1 \\ 1 & 1 & -1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & -1 \end{vmatrix}$$

$$=\frac{\Pi a^1}{n-1}\begin{vmatrix} -\sum\frac{1}{a_1} & \frac{1}{a_1} & \frac{1}{a_2} & \dots & \frac{1}{a_{n+1}} \\ 0 & -1 & 1 & \dots & 1 \\ 0 & 1 & -1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 1 & \dots & -1 \end{vmatrix}$$

obtained by subtracting all the other columns from $n-1$ times the first.

$$=\frac{-\Pi a_1\Sigma\frac{1}{a_1}}{n-1}\cdot\Delta_{n+1}$$

$$=-(-2)^n\Pi a_1\sum\frac{1}{a_1}.$$

$$D'_{n+1}=\begin{vmatrix} 0 & a_1+a_2 & a_1+a_3 & \dots & a_1+a_{n+1} \\ a_1+a_2 & 0 & a_2+a_2 & \dots & a_2+a_{n+1} \\ a_1+a_3 & a_2+a_3 & 0 & \dots & a_3+a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_1+a_{n+1} & a_2+a_{n+1} & a_3+a_{n+1} & \dots & 0 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & a_1+a_2 & a_1+a_3 & \dots & a_1+a_{n+1} \\ 0 & a_1+a_2 & 0 & a_2+a_3 & \dots & a_2+a_{n+1} \\ 0 & a_1+a_3 & a_2+a_3 & 0 & \dots & a_3+a_{n+1} \\ & \dots & \dots & \dots & \dots & \dots \\ 0 & a_1+a_{n+1} & a_2+a_{n+1} & a_3+a_{n-1} & \dots & 0 \end{vmatrix}$$

obtained by adding one row and one column more, making it a determinant of the $(n+2)^{th}$ order.

$$=-\frac{1}{2}\begin{vmatrix} (n-1)1+0 & 1 & 1 & \dots & 1 \\ (n-1)a_1+\Sigma a_1 & 0 & a_1+a_3 & \dots & a_1+a_{n+1} \\ (n-1)a_2+\Sigma a_1 & a_1+a_2 & 0 & \dots & a_2+a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ (n-1)a_{n+1}+\Sigma a_1 & a_1+a_{n+1} & a_2+a_{n+1} & \dots & 0 \end{vmatrix}$$

obtained by adding all the other columns to (-2) times the first

$$=-\frac{n-1}{2}\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & 0 & a_1+a_2 & \dots & a_3+a_{n+1} \\ a_2 & a_1+a_2 & 0 & \dots & a_2+a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n+1} & a_1+a_{n+1} & a_2+a_{n+1} & \dots & 0 \end{vmatrix}$$

$$-\frac{\Sigma a_1}{2}\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & a_1+a_2 & \dots & a_1+a_{n+1} \\ 1 & a_1+a_2 & 0 & \dots & a_2+a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1+a_{n+1} & a_2+a_{n+1} & \dots & 0 \end{vmatrix}$$

obtained by splitting up into two determinants.

$$=-\frac{n-1}{2}\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -a_1 & a_2 & \dots & a_{n+1} \\ 0 & a_1 & -a_2 & \dots & a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a_1 & a_2 & \dots & -a_{n+1} \end{vmatrix}-\frac{\Sigma a_1}{2}D_{n+2}$$

$$=-(n-)\frac{\Pi a_1}{2}\begin{vmatrix} -1 & 1 & 1 & \dots & 1 \\ 1 & -1 & 1 & \dots & \\ 1 & 1 & -1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 \end{vmatrix}-\frac{\Sigma a_1}{2}D_{n+2}$$

$$=-(n-1)\frac{\Pi a_1}{2}\Delta_{n+1}-\frac{\Sigma a_1}{2}D_{n+2}$$

$$=-\frac{\Pi a_1}{2}(n-1)(-2)^n(n-1)+\frac{\Sigma a_1}{2}(-2)^n\Pi a_1\Sigma\frac{1}{a_1}$$

$$=\Pi a_1(-2)^{n-1}(n-1)^2-\Pi a_1\Sigma a_1\Sigma\frac{1}{a_1}(-2)^{n-1}$$

$$=(-2)^{n-1}\Pi a_1\left[(n-1)^2-\Sigma a_1\Sigma\frac{1}{a_1}\right].$$

Question 658.

(K. J. Sanjana, M.A.) :—Prove the identity

$$2[ax+by+(a+b)(x+y)]^3-6[ax+by+(a+b)(x+y)]$$
$$[a^2+ab+b^2]\,[x^2+xy+y^2]-$$
$$27\,ab\,xy\,(a+b)\,(x+y)=(a-b)\,(x-y)\,(2a+b)\,(2b+a)\,(2x+y)(2y+x),$$

and deduce the theorem of Cayley, that

$$ax+by+cz,\ bx+cy+az,\ cx+ay+bz$$

are the roots of the equation

$$4t^3-3(a^2+b^2+c^2)(x^2+y^2+z^2)\,t-54abc\,xyz$$
$$-2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)=0,$$

when $a+b+c$ and $x+y+z$ vanish.

Solution by K. B. Madhava.

The expression on the left hand side of the identity is symmetrical separately in a, b and x, y and between themselves and is homogeneous in them of the third degree.

We see that $a-b$ is a factor, because the first member becomes identically zero when in it we put $a=b$.

Again, since we can write the expression in the form

$$2\,\{\,(2a+b)x+(a+2b)y\,\}^3-6\,\{\,(2a+b)x+(a+2b)y\,\}\times$$
$$\{\,\tfrac{1}{2}\,(2a+b)(a+2b)-\tfrac{3}{2}ab\,\}\,(x^2+y^2+z^2)-27ab\,xy\,\,(2a+b-a)(x+y)$$

we notice that $2a+b$ is a factor.

Similarly $a+2b$ is a factor.

From this we can also infer that $x-y$, $2x+y$, $2y+x$ are also factors. We can also satisfy ourselves that the constant multiplier is unity.

Hence the identity.

To deduce the second result, we have $a+b+c=0$ and $x+y+z=0$, $(a+b)(x+y)=cz$; $2a+b=a-c$; $2b+a=b-c$; $a^2+ab+c^2=\frac{1}{2}(a^2+b^2+c^2)$,&c.

Inserting these, we have

$$4(ax+by+cz)^3-3(a^2+b^2+c^2)\,(x^2+y^2+z^2)\,\,(ax+by+cz)-$$
$$54\,abc\,xyz-\ldots=0$$

which shows that $ax+by+cz$, and on account of symmetry $bx+cy+az$, $cx+ay+bz$ also, are the three roots of the given cubic in t.

Question 659.

(M. Bhimasena Rao) :—Show that

$$\text{(i)}\ (1+e^{-\pi})(1+e^{-2\pi})(1+e^{-3\pi})\ldots=e^{\frac{\pi}{24}}\left(\frac{1}{2}\right)^{\frac{1}{8}}.$$

$$\text{(ii}\ (1+e^{-\pi\sqrt{2}})(1+e^{-2\pi\sqrt{2}})(1+e^{3\pi\sqrt{2}})\ldots=e^{\frac{\pi\sqrt{2}}{24}}\left(\frac{1}{2}\right)^{\frac{1}{4}}.$$

Solution by K. B. Madhava.

Without using Elliptic Functions these results may be got from the simpler theory of Infinite Products as in Bromwich, Chap. VI, Ex. (14—20).

Borrowing his notation

$$q_1=\Pi(1+q^{2n})\ ;\ q_2=\Pi(1+q^{2n-1})\ ;\ q_3=\Pi(1-q^{2n-1})$$

we have $q_1q_2q_3=1$ (Ex. 14), $q_2^8=q_3^8+16qq_1^8$ (Ex. 20).

Now

$$q_2=e^{-\frac{\pi}{24}}\,2^{\frac{1}{4}}\ \text{(Q. 608)}$$

$$q_2^8=q_1^{-8}q_2^{-8}+16q\,q_1^8$$

$$4e^{-\frac{1}{3}\pi}=\tfrac{1}{4}e^{\frac{1}{3}\pi}q_1-8+16q_1^8e^{-\pi}$$

$$\therefore\ (8e^{-\frac{2}{3}\pi}q_1^8-1)^2=0,\ i.e.\ q_1=e\,e^{\frac{1}{12}\pi}\,2^{-\frac{3}{8}}\ ;\ q_1q_2=e^{\frac{1}{24}\pi}\,2^{-\frac{1}{8}}$$

which is result (i).

Again $q_2=e^{-\frac{\pi\sqrt{2}}{24}}.2^{\frac{1}{8}}(1+\sqrt{2})^{\frac{1}{8}}$

and the above equation for q_1^8 solved will give q_1q_2 in this case

$$=e^{\frac{\pi\sqrt{2}}{24}}\left(\frac{1}{2}\right)^{\frac{1}{4}}$$

Question 663.

(S. Krishnaswami Aiyangar, B. A.) :—Establish the following relation with respect to Bernoulli's numbers:

$$\frac{n}{2}=c_2B_1-c_4B_3\cdot4+c_6B_5\cdot4^2\ldots(-)^{n-1}c_{2n}B_{2n-1}4^{n-1},$$

where $c_r={}_{2n+1}C_r$.

Solution by K. B. Madhava and J. C. Swaminarayan, M.A.

We have

$$1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\ldots\ldots(-)^n\frac{\theta^{2n}}{(2n)!}\ldots=\cos\theta=\sin\theta\times\cot\theta$$

$$=\left[\theta-\frac{\theta^3}{3!}\ldots(-)^n\frac{\theta^{2n+1}}{(2n+1)!}\ldots\right]\left[\frac{1}{\theta}-\frac{2^2B_1}{2!}\theta-\frac{2^4B_3}{4!}\theta^3\ldots-\frac{2^{2n}B^{2n-1}}{(2n)!}\theta_{2n-1}\ldots\right]$$

Equating the coefficient of θ^{2n}, we have

$$\frac{1}{(2n)!}-\frac{1}{(2n+1)!}=\frac{2^2B_1}{2!\,(2n-1)!}-\frac{2^4B_3}{(4!\,2n-3)!}\ldots(-)^{n-1}\frac{2^{2n}B_{2n-1}}{2n!}$$

i.e. $$2n=2^2.c_2B_1-2^4c_4B_3+\ldots(-)^{n-1}2^{2n}c_{2n}B_{2n-1}$$

i.e. $$\frac{n}{2}=c_2B_1-c_4B_3\cdot4+c_6B_5\cdot4^2\ldots(-)^{n-1}c_{2n}B_{2n-1}4^{n-1}.$$

Question 664.

(S. KRISHNASWAMI AIYANGAR, B.A.) :—Prove that

$$\sum_{1}^{\infty}\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^2\cdot\frac{1}{n}=\frac{16}{\pi}-4.$$

Solution by J. C. Swaminarayan, M.A.

We have to prove that

$$\sum_{1}\left\{\frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}\right\}^2\cdot\frac{1}{n}=\sum_{1}^{\infty}\int_{0}^{1}\int_{0}^{1}\frac{x^n y^n\,dx\,dy}{n\sqrt{(1-x)(1-y)}}=16-4\pi.$$

Now $\displaystyle\sum_{1}^{\infty}\int_{0}^{1}\int_{0}^{1}\frac{x^n y^n\,dx\,dy}{n\sqrt{(1-x)(1-y)}}\cdot\int_{0}^{1}\int_{0}^{1}\frac{-\log(1-xy)}{\sqrt{(1-x)(1-y)}}\,dx\,dy$

$$=\int_{0}^{1}\frac{dx}{\sqrt{1-x}}\left[\left\{2\sqrt{1-y}\log(1-xy)\right\}_{0}^{1}+2\int_{0}^{1}\frac{x\sqrt{1-y}}{1-xy}dy\right]$$

$$=\int_{0}^{1}\frac{2dx}{\sqrt{1-x}}\left[2\int_{0}^{1}\left\{1-\frac{1-x}{1-x+xz^2}\right\}dz\right],\text{ where }1-y=z^2$$

$$=4\int_{0}^{1}\left[1-\sqrt{\frac{1-x}{x}}\tan^{-1}\left(\sqrt{\frac{x}{1-x}}\right)\right]\frac{dx}{\sqrt{1-x}}$$

$$=4\int_{0}^{1}\frac{dx}{\sqrt{1-x}}-4\int_{0}^{1}\frac{\tan^{-1}\sqrt{\frac{x}{1-x}}}{\sqrt{x}}dx$$

$$=\left[-8\sqrt{1-x}\right]_{0}^{1}-4\int_{0}^{\frac{\pi}{2}}\frac{\tan^{-1}(\tan\theta)}{\sin\theta}\times 2\sin\theta\cos\theta\,d\theta$$

[where $x=\sin^2\theta$]

$$=8-8\int_{0}^{\frac{\pi}{2}}\theta\cos\theta\,d\theta$$

$$=8-8\left[\theta\sin\theta+\cos\theta\right]_{0}^{\frac{\pi}{2}}$$

$$=8-8\left[\frac{\pi}{2}-1\right]$$

$$=8+8-4\pi=16-4\pi.$$

Question 672.

(S. KRISHNASWAMI AIYANGAR, B.A.) :—The vertex A of the triangle of reference lies without, on, or within the director circle of the maximum inscribed ellipse according as the angle A is acute, right or obtuse.

Remarks and Solution (1) *by K. B. Madhava,* (2) *by Narasinga Rao, B.A.*

(1) That if the vertex A lay on the director circle of the maximum inscribed ellipse (in fact of any inscribed conic) A should be a right angle is obvious and needs no proof.

Also from the elementary properties of the figure it is clear that the squares on the tangents from the vertices to the director circle are

$$\tfrac{1}{3}\, bc \cos A\,;\ \tfrac{1}{3}\, ca \cos B\,;\ \tfrac{1}{3}\, ab \cos C$$

so that the equation of the director circle (in areals) is

$$a^2yz+b^2zx+c^2xy=\tfrac{1}{3}(x+y+z)(bc \cos A\, x+ca \cos B y+ab \cos C z).$$

Thus the power of the vertex A is $\frac{1}{3}\, bc \cos A$, which is positive, zero, or negative according as A is acute, right or obtuse. Hence the theorem.

The theorem is indeed true for any inscribed conic as is clear from the fact (See Asquith, P. 421, Ex. 33) that if in areal tangentials the equation of any inscribed conic be $fmn+gnl+hlm=0$, the equation of the director circle is

$$2(f+g+h)(a^2yz+b^2zx+c^2xy)=$$
$$(x+y+z)\,\{\,(b^2+c^2-a^2)fx+(c^2+a^2-b^2)gy+(a^2+b^2-c^2)hz\,\}\,.$$

(2) Let α be the angle between the tangents drawn from any point (x, y) to the ellipse $S=0$.

Then α is a function of x and y, say $f\,(x, y)$.

Consider now the loci $\alpha=$ constant.

We have the following results respecting these :—

(i) These loci are concentric and coaxial ovals.

(ii) If P be a point on a fixed line through the centre O and distant r from it, then $\alpha=\phi(r)$ is a continually increasing or continually decreasing function of r, for, a stationary value of $\phi(r)$ corresponds to OP being a tangent at P to some curve of the system which is impossible. Also $\phi(r)$ vanishes for large values of r.

Hence $\phi(r)$ is a decreasing function of r.

(iii) All these loci are closed curves and no two can intersect in a real point ; for, if they do, the " α " of the common point will have two different values. Hence they lie one within the other. For values of α very near π the locus approximates to $S=0$ while for very small values of α, it tends to the line at infinity.

(iv) If P lies between the curves $\alpha=\alpha_1$ and $\alpha=\alpha_2$ then Pα lies between α_1 and α_2.

Let OP cut the two loci S_1 and S_2 in P_1 and P_2. Then P lies between P_1 and P_2 and therefore OP lies between OP_1 and OP_2. But $\phi(r)$ is monotonic.

$$\therefore \qquad \phi(r_2) < \phi(OP) < \phi(r_1)$$

$$i.e. \qquad \alpha_2 < \alpha < \alpha_1.$$

The converse proposition that if $\alpha<\alpha_1$, P lies without S_1, is proved on the same lines.

The given question is a particular case of this, for when $\alpha=\frac{1}{2}\pi$, Sα is the director circle. Hence if a Δ circumscribes an Ellipse, any vertex will lie within, on, or without the director ⊙ according as it is obtuse, right, or acute. This is true of a Δ and any inscribed Ellipse and in particular, of its maximum inscribed ellipse.

Question 675.

(K. Appukuttan Erady, M.A.) :—Shew that

$$\text{(i)} \int_0^{\pi} \frac{e^{b\cos\theta}\cos(b\sin\theta)}{1-2a\cos\theta+a^2}d\theta=\frac{\pi e^{ab}}{1-a^2};$$

$$\text{(ii)} \int_0^{\pi} \frac{\log(1-2b\cos\theta+b^2)}{1-2a\cos\theta+a^2}d\theta=\frac{2\pi}{1-a^2}\log(1-ab).$$

(1) *Solution by J. M. Bose, M.A, B.Sc.*

$$\text{(i)} \int_0^{\pi} \frac{e^{b\cos\theta}\cos(b\sin\theta)}{(1-2a\cos\theta+a^2)}\,d\theta.$$

Let

$$I=\int_0^{\pi} \frac{e^{b\cos\theta}\cos(b\sin\theta)}{1-2a\cos\theta+a^2}\,d\theta$$

$$J=\int_0^{\pi} \frac{e^{b\cos\theta}\sin(b\sin\theta)}{1-2a\cos\theta+a^2}\,d\theta;$$

so that

$$I+iJ=\int_0^{\pi} \frac{e^{be^{i\theta}}}{1-a(e^{i\theta}+e^{-i\theta})+a^2}\,d\theta$$

$$=\int_C \frac{ie^{bz}dz}{az^2-(1+a^2)z+a}=i\int_C \frac{e^{bz}dz}{(z-a)(az-1)}, \text{ putting } z=e^{i\theta}$$

where the integral is taken round a circle of unit radius in the z-plane.

The poles of the function are a and $\frac{1}{a}$.

(1) Let $a<1$, in which case the only pole within the contour is $z=a$, and the residue at this pole is $\frac{e^{ab}}{a^2-1}$.

$$\therefore \quad \mathrm{I} + i\,\mathrm{J} = \tfrac{1}{2}\left\{ 2\pi i \times \frac{ie^{ab}}{a^2-1} \right\} = \frac{\pi e^{ab}}{1-a^2}.$$

Similarly, (2) if $a>1$, then the pole inside the contour is $1/a$, and

$$\mathrm{I} + i\,\mathrm{J} = \frac{\pi e^{ba}}{a^2-1},$$

so that

$$\mathrm{I} = \frac{\pi e^{ab}}{1-a^2} \text{ and } \mathrm{J}=0.$$

(ii) $$\int_0^\pi \frac{\log(1-2b\cos\theta+b^2)}{(1-2a\cos\theta+a^2)}\,d\theta.$$

If we differentiate with respect to b

$$\frac{\partial \mathrm{I}}{\partial b} = \int_0^\pi \frac{2(b-\cos\theta)d\theta}{(1-2b\cos b+b^2)(1-2a\cos\theta+a^2)}$$

$$= \frac{1}{(1-ab)(b-a)} \int_0^\pi \left[\frac{a^2-2ab+1}{1-2a\cos\theta+a^2} - \frac{1-b^2}{1-2b\cos\theta+b^2} \right] d\theta.$$

Integrating each term separately we can get the required result, but the following method is more interesting.

$2\cos\theta = e^{i\theta} + e^{-i\theta}$ and $z=e^{i\theta}$ so that

$$\frac{\partial \mathrm{I}}{\partial b} = i\int_0^\pi \frac{z^2-2bz+1}{(bz-1)(z-b)(az-1)(z-a)}\,dz,$$

where the integral is to be taken round a circle of unit radius in the z-plane.

If $a<1$, $b<1$, then the two poles within the contour are a and b. The residues at these poles are

$$\frac{a^2-2ab+1}{(ab-1)(a-b)(a^2-1)} \text{ and } \frac{1-b^2}{(b^2-1)(ab-1)(b-1)} \text{ respectively;}$$

so that $$\frac{\partial \mathrm{I}}{\partial b} = \tfrac{1}{2}.\ 2\pi i \times \text{sum of the residues}$$

$$= \frac{-2a\pi}{(1-ab)(a^2-1)},$$

which gives the required result on integration.

Let $a>1$, $b>1$ then the poles are $1/a$ and $1/b$; and the residues are

$$\frac{ab(1-2ab+a^2)}{(b-a)(1-ab)(1-a^2)},\ \frac{ab(1-b^2)}{(a-b)(1-ab)};$$

so that

$$I=\frac{2\pi}{a^2-1}\log(ab-1).$$

(2) *Solution by A. Narasinga Rao, B.A.*

Poisson's formula states

$$\int_0^\pi \left[\frac{f(a+e^{ix})+f(a+e^{-ix})}{1-2c\cos x+c^2}\right]dx=\frac{2\pi}{1-c^2}f(a+c)$$

(Carr's *Synopsis*, 2702).

(i) Put $a=0$, $f(\Theta)=e^{b\Theta}$ and $c=a$. Then

$$\int_0^\pi \frac{e^{be^{ix}}+e^{be^{-ix}}}{1-2a\cos x+a^2}dx=\frac{2\pi}{1-a^2}e^{ab}.$$

i.e.
$$\int_0^\pi \frac{e^{b\cos x}\left(e^{ib\sin x}+e^{-ib\sin x}\right)}{1-2a\cos x+a^2}dx=\frac{2\pi e^{ab}}{1-a^2}$$

i.e.
$$\int_0^\pi \frac{e^{b\cos x}\cos(b\sin x)}{1-2a\cos x+a^2}dx=\frac{\pi e^{ab}}{1-a^2}.$$

(ii) Put $a=0$, $f(\Theta)=\log(1-b\Theta)$ and $c=a$.

$$\int_0^\pi \frac{\log(1-be^{ix})+\log(1-be^{-ix})}{1-2a\cos x+a^2}dx=\frac{2\pi}{(1-a^2)}\log(1-ab).$$

i.e.
$$\int_0^\pi \frac{\log(1-2b\cos x+b^2)}{1-2a\cos x+a^2}dx=\frac{2\pi}{1-a^2}\log(1-ab).$$

(3) *Solution by D. Krishna Murti.*

Let the integral (i) be denoted by w.
We can shew that

$$(1-2a\cos\Theta+a^2)^{-1}=\frac{1}{1-a^2}+\sum_{r=1}^{\infty}\frac{2a^r\cos r\Theta}{1-a^2}\quad \ldots\quad \ldots\quad \mathbf{A}$$

and that

$$e^{b\cos\Theta}\cos(b\sin\Theta)=\sum_{n=0}^{\infty}\frac{b^n\cos n\Theta}{n!}\quad \ldots\quad \ldots\quad \mathbf{B}$$

Consider the integral

$$w_r = \int_0^{\pi} e^{b\cos\theta} \cos(b\sin\theta)\cos r\theta\, d\theta.$$

With the aid of B, we get $w_r = \dfrac{\pi b^r}{2.r!}$

Now

$$w = \frac{1}{1-a^2}\int_0^{\pi} e^{b\cos\theta}\cos(b\sin\theta)\,d\theta + \sum_{r=1}^{\infty}\frac{2a^r}{1-a^2}.\, w_r \text{ (from A)}$$

i.e.

$$w = \frac{\pi}{1-a^2} + \frac{\pi}{1-a^2}\sum_{r=1}^{\infty}\frac{a^r b^r}{r!}$$

$$= \frac{\pi e^{ab}}{1-a^2}.$$

(ii) $$\int_0^{\pi}\frac{\log(1-2b\cos\theta+b^2)}{1-2a\cos\theta+a^2}.\, d\theta = \frac{2\pi}{1-a^2}\log(1-ab).$$

We have

$$I = \int_0^{\pi}\frac{\cos r\theta . d\theta}{1-2a\cos\theta+a^2} = \frac{\pi a^r}{1-a^2}, \text{ } r \text{ being an integer.}$$

For,

$$I = \frac{1}{1-a^2}\int_0^{\pi}\cos r\theta\, d\theta + \sum_{k=1}^{\infty}\int_0^{\pi}\frac{2a^k\cos k\theta\cos r\theta . d\theta}{1-a^2}$$

$$= \frac{2a^r}{1-a^2}\int_0^{\pi}\cos^2 r\theta . d\theta = \frac{\pi a^r}{1-a^2}.$$

Now let u denote the given integral. Then

$$\log(1-2b\cos\theta+b^2) = -2\sum_{n=1}^{\infty}\frac{b^n\cos n\theta}{n}.$$

$$\therefore \quad u = -2\sum_{n=1}^{\infty}\int_0^{\pi}\frac{b^n\cos n\theta}{n(1-2a\cos\theta+a^2)}.\, d\theta$$

$$= -2\sum_{n=1}^{\infty}\frac{\pi a^n b^n}{n(1-a^2)}$$

$$= \frac{2\pi}{(1-a^2)}\log(1-ab).$$

Question 689.

(J. C. SWAMINARAYAN, M.A.) :—If $S_m = 1^m + 2^m + 3^m + \dots n^m$, prove that

$$7S_6 + 5\,S_4 = 12\,S_1^2 S_2.$$

Solution (1) *by K. B. Madhava, R. Srinivasan, M.A. and S. V. Venkataraya Sastry, M.A., L.T.*; (2) *by R. D. Karve.*

(1) We know

$$S_m = \frac{n^{m+1}}{m+1} + \frac{1}{2}n^m + B_1\frac{m}{2\,!}.n^{m-1} - B_3\frac{m(m-1)(m-2)}{4\,!}.n^{m-3} + \dots$$

where B_1, B_3... are Bernoulli's numbers. (Hall and Knight's *Higher Algebra*, § 406.)

Hence

$$S_1 = \tfrac{1}{2}n\,(n+1),$$
$$S_2 = \tfrac{1}{6}n(n+1)(2n+1),$$
$$S_4 = \frac{n^5}{5} + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n,$$
$$S_6 = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}.$$

Making these substitutions we get the required result.

(2) We have

$$(2n+1)^7 - (2n-1)^7 = 896n^6 + 1120n^4 + 168n^2 + 2$$
$$(2n-1)^7 - (2n-3)^7 = 896(n-1)^6 + 1120(n-1)^4 + 168(n-1)^2) + 2$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$3^7 - 1^7 = 896.1^6 + 1120.1^4 + 168.1^2 + 2$$
$$\therefore \quad (2n+1)^7 - 1 = 896.S_6 + 1120.S_4 + 168S_2 + 2n,$$

giving

$$224S_6 + 280S_4 + 42S_2$$
$$= n(2n+1)(16n^5 + 48n^4 + 60n^3 + 40n^2 + 15n + 3).$$

Similarly,

$$120S_4 + 60S_2 = 6n(2n+1)(2n^3 + 4n^2 + 3n + 1).$$

Subtracting, we have

$$32(7S_6 + 5S_4) = n(2n+1)(16n^5 + 48n^4 + 48n^3 + 16n^2 - 3n - 3) + 18S_2,$$
$$= n(n+1)(2n+1)(16n^4 + 32n^3 + 16n^2 - 3) + 18S_2$$
$$= 6S_2 \cdot 16n^2(n+1)^2$$
$$= 6S_2 \cdot 16 \cdot 4S_1^2.$$

Thus

$$7S_6 + 5S_4 = 12S_2 \cdot S_1^2.$$

Question 690.

(J. C. SWAMINARAYAN, M.A.) :—If S_r denotes the sum of the reciprocals of the first r odd natural numbers, prove the following relations

(i) $$\frac{S_1}{3.4}-\frac{S_1}{3\cdot4}+\frac{S_2}{5\cdot6}-\frac{S_4}{7\cdot8}+\ldots\ldots$$
$$=-\frac{\pi^2}{32}+\frac{\pi}{8}\log 2+\frac{1}{2}\left\{1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots\right\}$$

(ii) $$\frac{1}{1\cdot2}-\frac{S_1}{3\cdot4}+\frac{S_2}{5\cdot6}-\frac{S_3}{7\cdot8}+\ldots\ldots$$
$$=-\frac{\pi^2}{32}+\frac{\pi}{4}+\left(\frac{\pi}{8}-\frac{1}{2}\right)\log 2+\frac{1}{2}\left\{\frac{1}{3^3}-\frac{1}{5^2}+\frac{1}{2^2}-\ldots\right\}.$$

Solution by Martyn M. Thomas, K. B. Madhava and K. R. Rama Iyer.

Now $$\frac{\tan^{-1}x}{1+x^2}=\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\ldots\right)\left(1-x^2+x^4-x^6+\ldots\ldots\right)$$
$$=x-x^3\left(1+\frac{1}{3}\right)+x^5\left(1+\frac{1}{3}+\frac{1}{5}\right)-x^7\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)+$$
$$=x\,S_1-x^3\,S_2+x^5\,S_3-x^7\,S_4+x^9\,S_5-\ldots \qquad \ldots \quad (1)$$

$$\therefore \quad \frac{1}{2}\frac{d}{dx}\{(\tan^{-1}x)^2\}=xS_1-x^3S_2+x^5S_3-x^7S_4+\ldots\ldots$$

Integrating the relation, we have

$$\tfrac{1}{2}\{\tan^{-1}x\}^2=\frac{x^2}{2}S_1-\frac{x^4}{4}S_2+\frac{x^6}{6}S_3-\frac{x^8}{8}S_4+\ldots$$

Dividing by x^2 and integrating between the limits 0 and 1,

$$\frac{1}{2}\int_0^1\frac{(\tan^{-1}x)^2}{x^2}dx=\left[\frac{x}{1\cdot2}S_1-\frac{x^3}{3\cdot4}S_2+\frac{x^5}{5\cdot6}S_3-\frac{x^7}{7\cdot8}S_4+\ldots\right]_0^1$$
$$=\frac{S_1}{1\cdot2}-\frac{S_2}{3\cdot4}+\frac{S_3}{5\cdot6}-\frac{S_4}{7\cdot8}+\ldots$$

Hence the given series (I)

$$=\tfrac{1}{2}\int_0^1\frac{(\tan^{-1}x)^2}{x^2}dx$$

$$=\tfrac{1}{2}\int_0^{\frac{\pi}{4}}\theta^2\operatorname{cosec}^2\theta\,d\theta$$

$$=\tfrac{1}{2}[-\theta^2\cot\theta]+\int_0^{\frac{\pi}{4}}\theta\cot\theta\,d\theta$$

$$=-\frac{\pi^2}{32}+\Big[\theta\log\sin\theta\Big]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\log\sin\theta\,d\theta$$

$$=-\frac{\pi^2}{32}+\frac{\pi}{4}\log\frac{1}{\sqrt{2}}-\int_0^{\frac{\pi}{4}}\Big[-\log 2-$$

$$\{\cos 2\theta+\tfrac{1}{2}\cos 4\theta+\tfrac{1}{3}\cos 6\theta+\dots\}\Big]d\theta$$

$$=-\frac{\pi^2}{32}-\frac{\pi}{8}\log 2+\frac{\pi}{4}\log 2+\Big[\frac{\sin 2\theta}{2}+\tfrac{1}{2}.\frac{\sin 4\theta}{4}+\tfrac{1}{3}.\frac{\sin 6\theta}{6}+\dots\Big]_0^{\frac{\pi}{4}}$$

$$=-\frac{\pi^2}{32}+\frac{\pi}{8}\log 2+\tfrac{1}{2}\Big[1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots\Big]$$

Again multiply (1) throughout by x^2 and x and integrate between the limits 0 and 1 and take the difference. Then

$$\int_0^1\frac{x^2\tan^{-1}x}{1+x^2}\,dx-\int_0^1\frac{x\tan^{-1}x}{1+x^2}\,dx$$

$$=\Big[\frac{S_1}{4}-\frac{S_2}{6}+\frac{S_3}{8}-\frac{S_4}{10}+\dots\Big]-\Big[\frac{S_1}{3}-\frac{S_2}{5}+\frac{S_3}{7}-\frac{S_4}{9}+\dots\Big]$$

$$=-\frac{S_1}{3.4}+\frac{S_2}{5.6}-\frac{S_3}{7.8}+\dots\dots$$

But

$$\int_0^1\frac{x^2\tan^{-1}x}{1+x^2}\,dx=\int_0^{\frac{\pi}{4}}\theta\tan^2\theta\,d\theta$$

$$=\int_0^{\frac{1}{4}\pi}\theta\sec^2\theta\,d\theta-\tfrac{1}{2}.\frac{\pi^2}{16}$$

$$=\Big[\theta\tan\theta\Big]-\int_0^{\frac{1}{4}\pi}\tan\theta\,d\theta-\frac{\pi^2}{32}$$

$$=\frac{\pi}{4}+\log\cos\frac{\pi}{4}-\frac{\pi^2}{32}=\frac{\pi}{4}-\tfrac{1}{2}\log 2-\frac{\pi^2}{32}$$

and

$$\int_0^1\frac{x\tan^{-1}x}{1+x^2}\,dx=\int_0^{\frac{\pi}{4}}\theta\tan\theta=\Big[-\theta\log\cos\theta\Big]_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\log\cos\theta\,d\theta$$

$$=-\frac{\pi}{4}\log\frac{1}{\sqrt{2}}+\int_0^{\frac{\pi}{4}}\Big[-\log 2+$$

$$\{\cos 2\theta-\tfrac{1}{2}\cos 4\theta+\tfrac{1}{3}\cos 6\theta-\dots\}\Big]d\theta$$

$$=\frac{\pi}{8}\log 2-\frac{\pi}{4}\log 2+\tfrac{1}{2}\Big\{1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots\Big\}$$

Hence the series (ii)

$$=\frac{\pi}{4}-\frac{\pi^2}{32}+\Big(\frac{\pi}{8}-\tfrac{1}{2}\Big)\log 2+\tfrac{1}{2}\Big\{\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}-\dots\Big\}.$$

QUESTIONS FOR SOLUTION.

741. (Martyn M. Thomas) :—Two particles A and B describe coplanar closed orbits in such a manner that, at every instant, the line joining them represents, in magnitude and direction, the velocity of A in its orbit. Show that the area of the hodograph of A's orbit

$$=(1+n)\left\{ B+\frac{A}{n}-G\left(1+\frac{1}{n}\right)\right\}k^2,$$

where A,B,G represent the areas of the orbits described by the particles A, B and their centre of gravity, n is the ratio of the mass of B to that of A, and k a constant.

742. (Martyn M. Thomas) :—Show that a curve and all its pedals, positive and negative, have the same potential, at the pedal origin.

743. (N. Salva) :—Prove the following construction for the Feuerbach-point :

If AI meet the circumcircle in P and P′ be the reflection of P in BC, then the Feuerbach-point F is the reflection of D in IP′, where D is the point of contact of the incircle and BC.

744. (N. Salva) ;—If ABCD, AB′ CD′ be two harmonic ranges such that D′ is the mid-point of CD, prove that $BC^2=BB'.BD'$.

745. (V. Anantaraman) :—How can 49 diamonds, whose values are in A. P., be divided among 7 persons so that each may get the same number of diamonds, the total value of the diamonds got by each being the same ?

746. (Lakshmishankar N. Bhatt).—Prove that, n being an integer not less than 2,

$$\left\{\frac{1}{n-1}+\frac{1}{2}\cdot\frac{1}{3n-1}+\frac{1.3}{2.4}\,\frac{1}{5n-1}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7n-1}+\ldots ad\ inf.\right\}$$

$$\times\left\{\frac{1}{n+1}+\frac{1}{2}\cdot\frac{1}{3n+1}+\frac{1.3}{2\cdot4}\cdot\frac{1}{5n+1}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{7n+1}+\ldots ad\ inf.\right\}$$

$$=\frac{2}{(n-1)(n+1)}+\frac{2}{(3n-1)(3n+1)}+\frac{2}{(5n-1)(5n+1)}+\ldots ad\ inf.$$

747. (K. Appukuttan Erady, M. A.) :—(A, B, C, D, E) and (P, Q, R, S, T) are two sets of points in space ; prove that

$$\begin{vmatrix} AP^2, & AQ^2, & AR^2, & AS^2, & AT^2 \\ BP^2, & BQ^2, & BR^2, & BS^2, & BT^2 \\ CP^2, & CQ^2, & CR^2, & CS^2, & CT^2 \\ DP^2, & DQ^2, & DR^2, & DS^2, & DT^2 \\ EP^2, & EQ^2, & ER^2, & ES^2, & ET^2 \end{vmatrix} = 0,$$

provided a sphere passes through either set.

748. (K. Appukuttan Erady, M. A.) .—If a, b, c, d be the sides $2s$ the perimeter, and 2S, 2S′ the sums of pairs of opposite angles of a spherical quadrilateral on a sphere of unit radius, show that

$$\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}\cos\frac{d}{2}\cos^2\frac{1}{2}(S+S')$$
$$+\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}\sin\frac{d}{2}\sin^2\tfrac{1}{2}(S-S')$$
$$=\sin\tfrac{1}{2}(s-a)\sin\tfrac{1}{2}(s-b)\sin\tfrac{1}{2}(s-c)\sin\tfrac{1}{2}(s-d).$$

749. (S. Krishnaswami Aiyangar) :—If ρ be the radius of curvature of the curve $r^m = a^m \sin m\theta$, ρ_k the radius of curvature of the corresponding point of its kth negative pedal with regard to the pole, show that

$$(1-mk+m)^{m-1}a^{mk}\rho_k{}^{m-1}=(1-mk)^{m-1}\{(m+1)\rho\}^{mk+m-1}.$$

750. (S. Krishnaswami Aiyangar):—If $(c-z)\,x\sin B=(a-x)y\sin C=(b-y)\,z\sin A$, and $A+B+C=180^\circ$, show that

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}.$$

Give also a geometrical interpretation of the result.

751. (Selected) :—Prove that

$$\sum_{-p}^{p}\;\sum_{n=-p}^{p}\left\{\frac{2x+m+n}{(x+m)^2(x+n)^2}\right\}\to 0$$

when $p\to\infty$, provided that all terms for which $m=n$ are omitted from the summation. [Tripos : 1895].

752. (R. Srinivasan, M.A.) :—Four rods, AB, BC, CD, DA are smoothly jointed at the extremities so as to form a quadrilateral ABCD in a vertical plane. A is fixed and C is held at a given vertical height above A. If C is now released and motion ensue, discuss the motion immediately before and after the impact which takes place when BC and CD straighten into a line. (The rods are uniform and $AB+DA>BC+CD$).

753. (S. Ramanujan) :—If

$$\phi(x)=\tfrac{1}{2}\log 2\pi x-x+\int_1^x \frac{[t]}{t}\,dt,$$

where $[t]$ denotes the greatest integer in t, show that

$$\lim_{x\to\infty} x\,\phi(x)=\frac{1}{24};\quad \lim_{x\to\infty} x\,\phi(x)=-\frac{1}{12}.$$

754. (S. Ramanujan) :—Show that

$$\frac{e^x\Gamma(1+x)}{x^x\sqrt{\pi}}=\sqrt[6]{8x^3+4x^2+x+E},$$

where E lies between $\frac{1}{100}$ and $\frac{1}{30}$ for all positive values of x.

755. (S. Ramanujan) :—Let p be the perimeter and e the eccentricity of an ellipse whose centre is C, and let CA and CB be a semi-major and a semi-minor axis. From CA cut off CQ equal to CB and also produce AC to P making CP equal to CB. From A draw AN perpendicular to CA (in the direction of CB). From Q draw QM making with QA an angle equal to ϕ (which is to be determined) and meeting AN at M. Join PM and draw PN making with PM an angle equal to half of the angle APM and meeting AN at N. With P as centre and PA as radius describe a circle cutting PN at K and meeting PB produced at L. Then if

$$\frac{arc\ \mathrm{AL}}{arc\ \mathrm{AK}}=\frac{p}{4\mathrm{AN}}$$

trace the changes in ϕ when e varies from 0 to 1. In particular, show that $\phi=30°$, when $e=0$; $\phi\to 30°$, when $e\to 1$; $\phi=35°$, when $e=.99948$ nearly; ϕ assumes the minimum value of about $29°\ 58\tfrac{3}{4}'$, when e is about $\cdot 999886$; and ϕ assumes the maximum value of about $30°\ 44\tfrac{1}{4}'$ when e is about $\cdot 9589$.

756. (S. Narayana Aiyar, M. A.) :—If

$$A_n=\frac{(b-c)(b-cx)(b-cx^2)\ldots\ldots(b-cx^{n-1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^n)},$$

$$B_n=\frac{(c-a)(c-ax)(c-ax^2)\ldots\ldots(c-ax^{n-1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^n)}$$

and

$$C_n=\frac{(a-b)(a-bx)(a-bx^2)\ldots\ldots(a-bx^{n-1})}{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^n)},$$

show that

$$A_n+A_{n-1}(B_1+C_1)+A_{n-2}(B_2+B_1C_1+C_2)$$
$$+A_{n-3}(B_3+B_2C_1+B_1C_2+C_3)+\ldots\ldots\ldots=0.$$